ENGINEERING EXPERIMENT STATION
of the Georgia Institute of Technology
Atlanta, Georgia

A DESCRIPTION OF A NEAR EARTH SATELLITE ORBIT COMPUTATION PROGRAM USING OBLATE SPHEROIDAL COORDINATES

By

E. L. Davis and J. G. Stein

FINAL REPORT

PROJECT A-478

Contract No. NAS1-532

Prepared for
National Aeronautics and Space Administration
Langley Research Center
Langley Field, Virginia

May 1961

ABSTRACT

This report describes a computing procedure for obtaining the coordinates and velocity of a near earth satellite from measurements of its range and/or angular position in the sky. The procedure, first suggested by Vinti (1959), differs from other computing procedures in that the oblateness potential is included in the analytical solution of the equations of motion. This solution is expressed in terms of elliptic integrals of the first, second and third kinds. The modulus of the elliptic integrals is of the order of magnitude of the coefficient of the second term in the harmonic expansion of the earth's gravitational potential, so that the series expansions of the elliptic integrals converge rapidly. In the computing program coded from this procedure, the elliptic integrals are evaluated in subroutines which, from the viewpoint of programming, number of locations, and number of operations are entirely comparable to subroutines for the elementary functions. The procedure applies to near earth satellites for all values of eccentricity less than unity, and for all inclinations with the trivial exception that for polar orbits, the celestial longitude is indeterminate in the immediate vicinity of the celestial poles. Drag forces, solar radiation forces, etc., are included by means of a "variation of constants" method.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the contributions made by James R. Garrett and Joe W. Wray during the early phases of the work, and also the contribution made by Mrs. Jeanne M. Crawford during the final stages of report preparation.

A DESCRIPTION OF A NEAR EARTH SATELLITE ORBIT COMPUTATION PROGRAM USING OBLATE SPHEROIDAL COORDINATES

Respectfully submitted by:

Eugene L. Davis, Jr.
Project Director

Approved by:

Wm. F. Atchison, Head
Rich Electronic Computer Center

J. E. Boyd, Director
Engineering Experiment Station

TABLE OF CONTENTS

A DESCRIPTION OF A NEAR-EARTH SATELLITE ORBIT COMPUTATION PROGRAM USING OBLATE SPHEROIDAL COORDINATES

- o -

1. Introduction

The equations of motion of a near-earth satellite have received much attention in recent years. No solution of these equations involving a finite number of elementary functions, or integrals of elementary functions, has been found. Numerical integration of the equations presents serious practical difficulties because of the round-off error, the truncation error, and the problem of improving initial conditions from observations. The lack of an analytical solution, and the practical difficulties associated with direct numerical integration have led to the modification of older methods and to the development of newer approximate algebraic-numeric methods [see, for example, Ref. 2 - 7]. These approximate methods are based on the computation of small motions relative to a reference orbit. The reference orbit is obtained analytically; the small "residual" motions are computed by (a) expanding the disturbing functions in a Fourier series [Ref. 2, 3], or (b) by numerical integration [Ref. 4, 5]. Each of these methods requires a literal algebraic development. The complexity of the computational procedure, that is, the order of terms which must be retained in the algebraic development to insure accuracy, and the problems of computing time, computer size, and so on, is determined largely by the reference orbit selected. The path of an earth satellite for an arc of, say, one revolution is approximately Keplerian, so that at first sight the logical choice for the reference orbit is a Keplerian orbit. Nevertheless, the deviation of the actual path from a Keplerian orbit is great enough to introduce objectional complexities into a literal development.

A study of the computational problems associated with the orbits of artificial earth satellites over arcs of many revolutions leads to the conclusion that a reference orbit based on Keplerian motion is not the optimum one. A number of modified reference orbits have been devised [Ref. 5,6,7].

In an important paper, Sterne [Ref. 8] pointed out that by including a part of the oblateness potential in the Hamiltonian, an exact analytical solution of the Hamilton-Jacobi equation yielded a skewed, non-periodic reference orbit which was competitive in accuracy to a Keplerian orbit with first order perturbations. Garfinkle [Ref. 9] investigated the various forms of the solution of the Hamilton-Jacobi equation obtained by including part of the oblateness potential in the Hamiltonian. In these two papers, the Hamiltonian is expressed in spherical polar coordinates, and only part of the quadrupole moment term in the harmonic expansion of the earth's potential is included with the monopole term. Vinti [Ref. 1] demonstrated the existence of a rotationally symmetric solution of Laplace's equation in oblate spheroidal coordinates which (a) fits the experimentally determined values of the earth's gravitational potential within ± 0.2 parts per million, and which (b) is in a form which permits an exact analytical solution of the Hamilton-Jacobi equations. The gravitational force due to the earth's oblateness is the overwhelmingly important non-central force acting to disturb the path of motion of near-earth satellites from a Keplerian orbit, for all satellites which are not so near as to be brought down quickly by atmospheric resistance. Vinti's method makes it possible to incorporate this largest non-central force into the analytical solution in a relatively simple manner. The reference orbit should be competitive in accuracy with a Keplerian orbit with very high order perturbations; comparison of the results of Vinti's method, as formulated in this paper, with precise numerical integrations of the equations of motion, and the comparison of computed with actual observations on existing satellites over long arcs, confirm this. The remaining non-central forces are small, so that conventional perturbation procedures are conveniently applied.

The purpose of this paper is to give a formulation of Vinti's method applicable to the computation of satellite orbits. In this formulation, the kinetic equations of motion are transformed into standard elliptic integrals of the first, second and third kinds, with arguments analogous to the true anomaly and the argument of latitude in Keplerian motion. The two equations which replace Kepler's equation are numerically inverted to obtain the arguments at a given time. The formulation is well adapted to digital computer applications.[1] The elliptic integrals may be evaluated in subroutines which, from the viewpoint of programming, number of locations, and number of operations are no different from subroutines for elementary functions. The procedure requires significantly fewer computer internal storage locations, and requires much less computing time than a perturbed Keplerian motion procedure of comparable precision.

[1] Orbit computation programs based on the equations given in this paper have been coded at the Rich Electronic Computer Center, Georgia Institute of Technology, and at the Langley Research Center, NASA.

2. Vinti's Solution

Oblate spheroidal coordinates are defined by [Ref. 11, p. 662].

$$(1)\qquad \begin{aligned} x &= \sqrt{(\xi^2 + c^2)(1 - \eta^2)}\ \cos\varphi \\ y &= \sqrt{(\xi^2 + c^2)(1 - \eta^2)}\ \sin\varphi \\ z &= \xi\,\eta \end{aligned}$$

where x, y, z form a right-handed, non-rotating rectangular coordinate system, with the positive axis of z coinciding with the earth's north polar axis, x, y lying in the earth's equatorial plane, ξ, η, φ are oblate spheroidal coordinates, and c^2 is a constant adjusted to equal in magnitude the strength of the quadrupole term in the earth's gravitational field. Vinti demonstrated that the earth's gravitational potential could be approximated within ± 0.2 parts per million by a potential of the form

$$(2)\qquad V = \frac{-\xi}{\xi^2 + \eta^2 c^2}$$

This is in a form which permits separation of the Hamilton-Jacobi equation. The exact analytical solution of the Hamilton-Jacobi equation yields six kinetic equations of motion, in terms of six canonical constants α_i, β_i $(i = 1,2,3)$

$$(3)\qquad \begin{aligned} p_\xi &= \pm\frac{\sqrt{R(\xi)}}{\xi^2 + c^2} \\ p_\eta &= \pm\frac{\sqrt{R(\eta)}}{1 - \eta^2} \\ p_\varphi &= \beta_2 \\ t - \alpha_1 &= L_1(\xi) + L_2(\eta) \\ \varphi + \alpha_2 &= -M_1(\xi) + M_2(\eta) \\ \alpha_3 &= N_1(\xi) - N_2(\eta) \end{aligned}$$

where p_ξ, p_η, p_φ are the generalized momenta in the coordinates ξ, η, φ, and

$$L_1(\xi) = \int \frac{\xi^2 d\xi}{\pm\sqrt{R(\xi)}}, \qquad L_2(\eta) = c^2 \int \frac{\eta^2 d\eta}{\pm\sqrt{R(\eta)}},$$

$$M_1(\xi) = +\beta_2 c^2 \int \frac{d\xi}{\pm(\xi^2 + c^2)\sqrt{R(\xi)}}, \qquad M_2(\eta) = \beta_2 \int \frac{d\eta}{\pm(1 - \eta^2)\sqrt{R(\eta)}},$$

$$(4) \quad N_1(\xi) = \beta_3 \int \frac{d\xi}{\pm\sqrt{R(\xi)}}, \qquad N_2(\eta) = \beta_3 \int \frac{d\eta}{\pm\sqrt{R(\eta)}},$$

$$R(\xi) = \beta_2^2 c^2 + (\xi^2 + c^2)(2\beta_1\xi^2 + 2\xi - \beta_3^2),$$

$$R(\eta) = (\beta_3^2 + 2\beta_1\eta^2 c^2)(1 - \eta^2) - \beta_2^2 .$$

These are the equations given by Vinti. For details, the original paper by Vinti [Ref. 1] should be consulted. The two equations

$$t - \alpha_1 = L_1(\xi) + L_2(\eta),$$

$$\alpha_3 = N_1(\xi) - N_2(\eta),$$

are numerically inverted to obtain ξ and η. Since ξ, η, which are "amplitudes" of vibration, do not exhibit a one-to-one correspondence with the time, the actual inversion is to be done using angle variables which do exhibit a unique correspondence with the time. With the appropriate choice of transformations, the transformed equations exhibit a non-vanishing Jacobian, which insures the existence of a unique solution.

3. Reduction of the Elliptic Integrals to Standard Form

The integrals in (4) cannot be expressed by a finite number of elementary functions. Two choices of attack are possible: (a) the integrands may be expressed directly as a power series in terms of $O(c^2)$ and integrated, or (b) the integrals may be reduced to standard form. In the end, both methods are essentially equivalent, except that the second method takes advantage of the extensive literature on elliptic integrals, and is in a better form for programming into a digital computer.

For example, by means of a transformation of the form

$$\xi = a(1 - e \cos E)$$

with a and e appropriate constants, suggested by the analogous equation in Keplerian motion, the integrals in ξ may be expressed as a power series in the oblateness term c^2 with coefficients which are functions of the true anomaly. However, when the integrals are reduced to standard form, fewer terms in the series expansion are required for a given accuracy, and the integrals may be evaluated in subroutines which are not much different from subroutines for the elementary functions, in terms of number of locations and number of computational steps.

In the reduction of the elliptic integrals to standard form, a number of choices of transformations are available. The transformations which emphasize the parallel with Keplerian motion have been chosen. This has a number of advantages, the important ones being: (a) the "physics" of the familiar Keplerian motion is, in a sense, retained, since the equations of motion in the elliptic integral formulation differ from the equations of Keplerian motion only by terms of magnitude $O(c^2)$; (b) for far-distant satellites the equations of motion degenerate smoothly into equations of Keplerian motion; and (c) if products of the disturbing forces and c^2 can be neglected, the perturbation equations are identical in form to the perturbation equations of Keplerian motion.

a. Reduction of the Integrals $L_2(\eta)$, $M_2(\eta)$, $N_2(\eta)$

The integral $N_2(\eta)$ may be written

(5) $$N_2(\eta) = \frac{\beta_3}{\sqrt{-2\beta_1}\,c} \int \frac{d\eta}{\sqrt{(\eta^2 - \eta_1^2)(\eta - \eta_2^2)}} \; .$$

Here η_1^2 and η_2^2, the roots of $R(\eta)$, are

(6) $$\eta_{1,2}^2 = -\frac{1}{4\beta_1 c^2}\left[\beta_3^2 - 2\beta_1 c^2\right]\left[1 \pm \sqrt{1 + \frac{8(\beta_3^2 - \beta_2^2)\beta_1 c^2}{(\beta_3^2 - 2\beta_1 c^2)^2}}\right] .$$

For a bound particle, the total energy is negative, i.e., $\beta_1 < 0$ always; hence, $\eta_{1,2}^2$ are positive and real. In the limit as the oblateness vanishes

(7) $$\lim_{c^2 \to 0} \eta_{1,2}^2 \to \begin{cases} \eta_1^2 \to \infty \\ \\ \eta_2^2 \to \sin^2 i \end{cases}$$

where i is the inclination of the orbit. Let $\eta_2^2 < \eta_1^2$. Then the range of the variable of integration is

(8) $$\eta_2^2 > \eta^2 > 0 \; .$$

The limits (7) and the range of variables (8) suggest a transformation

(9) $$\eta = \eta_2 \sin u \; .$$

With this transformation, the integral becomes

(10) $$N_2(u) = \frac{\beta_3}{\sqrt{-2\beta_1}} \frac{1}{\eta_1 c} F(u, \chi_2) \; ,$$

where $F(u,\chi_2)$ is an elliptic integral of the first kind of argument u and modulus χ_2, with χ_2 defined by

$$\chi_2^2 = \frac{\eta_2^2}{\eta_1^2} \quad . \tag{11}$$

The argument u is analogous to the argument of latitude in Keplerian motion.

The remaining integrals in η become

$$\begin{aligned} L_2(u) &= \frac{\eta_1 c}{\sqrt{-2\beta_1}} \left[F(u, \chi_2) - E(u, \chi_2) \right] , \\ M_2(u) &= \frac{\beta_2}{\eta_1 c \sqrt{-2\beta_1}} \pi(u, \eta_2^2, \chi_2) , \end{aligned} \tag{12}$$

where $F(u, \chi_2)$, $E(u, \chi_2)$, $\pi(u, \eta_2^2, \chi_2)$ are elliptic integrals of the first, second and third kinds. In the limit as the oblateness vanishes

$$\begin{aligned} &\lim_{c^2 \to 0} L_2(u) \to 0 , \\ &\lim_{c^2 \to 0} M_2(u) \to \tan^{-1} (\cos i \tan u) . \end{aligned} \tag{13}$$

The last expression is the equation for the azimuthal angle from the ascending node to the object in Keplerian motion.

b. Reduction of the Integrals $L_1(\xi)$, $M_1(\xi)$, $N_1(\xi)$

The homographic transformation

$$\xi = \rho \frac{(1 + \ell \cos v)}{(1 + s_1 \cos v)} \tag{14}$$

emphasizes the parallel with Keplerian motion. The constants ρ, ℓ, and s_1 are chosen such that

$$\lim_{c^2 \to 0} \begin{cases} \rho \to a(1 - e^2) \\ \ell \to 0 \\ s_1 \to e \end{cases}$$

where a is the semi-major axis of a Keplerian orbit and e is the eccentricity. It is unnecessary to give the details of the transformation, since a step-by-step procedure for the reduction of elliptic integrals to standard form is given in standard texts [see, for example, Ref. 12, p. 180].

Write the polynomial $R(\xi)$, given in (4), as

$$R(\xi) = (2\beta_1)(\xi - \xi_1)(\xi - \xi_2)(\xi^2 + 2j\xi + k) ,$$

where ξ_1, ξ_2 are the largest, real, positive roots, with $\xi_1 > \xi_2$.

(15)

$$k = -c^2 \frac{(\beta_3^2 - \beta_2^2)}{2\beta_1\xi_1\xi_2} ,$$

$$j = \frac{1}{2\beta_1} + \frac{(\xi_1 + \xi_2)}{2} ,$$

$$A = \xi_1\xi_2 + j(\xi_1 + \xi_2) + k ,$$

$$B^2 = (k - j^2)(\xi_1 - \xi_2)^2 ,$$

$$C = A + \sqrt{A^2 + B^2} ,$$

$$s_1 = \frac{C(\xi_1 - \xi_2)(\xi_1 + j) - B^2}{C(\xi_1 - \xi_2)(\xi_1 + j) + C^2} ,$$

$$\ell = \frac{(\xi_1 + \xi_2)s_1 - (\xi_1 - \xi_2)}{(\xi_1 + \xi_2) - (\xi_1 - \xi_2)s_1} ,$$

$$X_1^2 = \frac{B^2}{B^2 + C^2} ,$$

$$Q = \frac{\sqrt{2C}}{\sqrt{C^2 + B^2}\,\sqrt{-2\beta_1}} ,$$

$$q = \rho(\ell - s_1) ,$$

$$s = \frac{s_1^2}{1 - s_1^2} .$$

The integral $N_1(\xi)$ becomes

(16) $$N_1(v) = Q\beta_3 F(v, \chi_1) .$$

The integral $L_1(\xi)$ becomes

(17) $$L_1(v) = \frac{\rho^2 Q}{s_1(s_1^2-1)} [(1+\ell^2)s_1 - 2\ell]F(v,\chi_1) + \frac{q^2 Q}{(s_1^2-1)[s_1^2(1-\chi_1^2) + \chi_1^2]} \left\{\frac{s_1 \sin v \sqrt{1-\chi_1^2 \sin^2 v}}{1 + s_1 \cos v} - E(v,\chi_1)\right\} - \rho Q q \left\{\frac{2\ell}{s_1^2} - \frac{q}{(s_1^2-1)[s_1^2(1-\chi_1^2) + \chi_1^2]} + \frac{2q}{\rho s_1^2(s_1^2-1)}\right\} \left\{\frac{1}{1-s_1^2}\pi(v,-s^2,\chi_1) - \frac{s_1}{1-s_1^2} \frac{1}{\sqrt{s^2+\chi_1^2}} \tan^{-1}\left(\frac{\sqrt{s^2+\chi_1^2}\sin v}{\sqrt{1-\chi_1^2 \sin v}}\right)\right\} .$$

The squared modulus χ_1^2 is a positive number of $O(c^2)$ for highly inclined orbits; it vanishes for an inclination angle approximately equal to 0.03 radians, and becomes slightly negative for equatorial orbits.

The integral $M_2(v)$ may be expressed in terms of elliptic integrals of the third kind with a complex parameter, but this form is poorly suited to computation. A better form is that obtained by a series expansion. Define the quantities

(18)
$$I_1 = \frac{Qs_1^2}{\rho^2\ell^2 + s_1^2 c^2}$$

$$R_1 = \frac{Q|q|}{\sqrt{(\rho^2\ell - s_1c^2)^2 + \rho^2c^2(\ell + s_1)^2}}$$

$$R_2 = \frac{\sqrt{(\rho^2\ell + s_1c^2) + \rho^2c^2(\ell - s_1)^2}}{\rho^2 + c^2}$$

$$\varphi_1 = \tan^{-1} \frac{\rho c(\ell + s_1)}{(\rho^2\ell - s_1c^2)}$$

$$\varphi_2 = \tan^{-1} \frac{\rho c(\ell - s_1)}{\rho^2\ell + s_1c^2}$$

The integral becomes

$$M_1(v) = \beta_2 c^2 I_1 + \beta_2 R_1 c \sum_{m=0}^{\infty} \binom{-1}{m} R_2^m \sin(\varphi_1 + m\varphi_2) C_m \tag{19}$$

where C_m is the integral

$$C_m = \int_0^v \frac{\cos^m v \, dv}{\sqrt{1 - \chi_1^2 \sin^2 v}} \quad . \tag{20}$$

This is a standard integral, but the usual literal form [see, for example, Ref. 13, p. 192] is poorly suited to numerical computation because of the presence of a small divisor. A better form is obtained by a Taylor's series expansion of the integrand. The results are

$$C_m = \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} (-\chi_1^2)^{\nu} J_{\nu m} \quad , \tag{21}$$

where $J_{\nu m}$ is given by the recursive formula

$$J_{\nu m} = \frac{1}{2\nu + m} [\sin^{2\nu+1} v \cos^{m-1} v + (m - 1) J_{\nu, m-2}] \quad , \tag{22}$$

with

$$\begin{aligned} J_{\nu 0} &= t_{2\nu}(v) \\ J_{\nu 1} &= \frac{1}{2\nu + 1} \sin^{2\nu+1} v \quad , \end{aligned} \tag{23}$$

and

$$\begin{aligned} t_{2\nu}(v) &= \int \sin^{2\nu} v \, dv = \frac{2\nu - 1}{2\nu} t_{2\nu-2}(v) - \frac{1}{2\nu} \sin^{2\nu-1} v \cos v \quad , \\ t_0(v) &= v \quad . \end{aligned} \tag{24}$$

The series expansion (21) converges rapidly. The first four terms yield an accuracy to eight decimal places.

4. Series Expansions for the Standard Elliptic Integrals

The elliptic integrals of the first, second and third kinds are (in Legendre's notation) defined by

$$F(\varphi, \chi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - \chi^2 \sin^2 \varphi}} ,$$

(25)
$$E(\varphi, \chi) = \int_0^{\varphi} \sqrt{1 - \chi^2 \sin^2 \varphi} \; d\varphi ,$$

$$\pi(\varphi, p^2, \chi) = \int_0^{\varphi} \frac{d\varphi}{(1 - p^2 \sin^2 \varphi) \sqrt{1 - \chi^2 \sin^2 \varphi}} ,$$

where χ is called the modulus, φ is called the argument, and p is called the parameter.

The integrals of the first and second kind may be defined by the series expressions

$$F(\varphi, \chi) = \sum_{m=o}^{\infty} \binom{-\frac{1}{2}}{m} (-\chi^2)^m \, t_{2m} (\varphi) ,$$

(26)
$$E(\varphi, \chi) = \sum_{m=o}^{\infty} \binom{\frac{1}{2}}{m} (-\chi^2)^m \, t_{2m} (\varphi) ,$$

where $\binom{n}{m}$ stands for the coefficients in the series expansion of

(27)
$$(1 + \chi)^n = \sum_{m=o}^{\infty} \binom{n}{m} \chi^m ,$$

and $t_{2m} (\varphi)$ is defined by equation (24).

The squared modulus χ^2 is $O(c^2)$, so that the power series expansion converges rapidly. The first four terms yield an accuracy to eight decimal digits.

The elliptic integral of the third kind is represented by two series, each applicable over different ranges of the parameter p. For p^2 small enough to insure convergence

(28) $$\pi(\varphi,\ p^2,\ \chi) = \sum_{m=o}^{\infty} \sum_{j=o}^{m} (p^2)^m\ t_{2m}\ (\varphi) \left(\frac{-\chi^2}{p^2}\right)^j ,$$

where $t_{2m}\ (\varphi)$ is defined in (24). The practical range of convergence is [Ref. 14].

$$|p^2| < \frac{1}{2}, \qquad \chi^2 < 1 .$$

When the magnitude of p^2 is greater than χ^2, the following series is used

(29) $$\pi(\varphi,\ p^2,\ \chi) = \sum_{m=o}^{\infty} \beta_m'\ \chi^{2m} ,$$

where the β_m' are given by [Ref. 13]:

(30) $$[2(m+1)p^2]\beta_{m+1}' = [2m(1+p^2)+1]\beta_m' + (1-2m)\ \beta_{m-1}' - (-1)^m\binom{\frac{1}{2}}{m} t_{2m}\ (\varphi) - (-1)^m\binom{-\frac{1}{2}}{m-1} \sin^{2m-1}\varphi \cos\varphi .$$

This last series is valid for values of p^2 in the range

$$|p^2| > 2\chi^2 ,$$

i.e., the series fails for nearly circular orbits and also for nearly equatorial orbits. A practical compromise is to use the series (29) where $|p^2|$ is greater than $2\chi^2$ (e.g., for eccentric and for non-equatorial orbits) and the series (28) when $|p^2|$ is less than or equal to $2\chi^2$ (e.g., for nearly circular and also for nearly equatorial orbits).

With the proper choice of series representations for the elliptic integral of the third kind, no singularities appear in the range

$$-\infty < p^2 < 1 .$$

This condition is violated for

$$s_1 \geq 1 ,$$

$$\eta_2^2 = 1 .$$

The first of these corresponds (approximately) to an eccentricity greater than or equal to unity, and is of no interest here. The second corresponds to a polar orbit. The difficulty appears in the first term in the series expansion for the elliptic integral of the third kind in the parameter η_2. This integral appears in the equation for α_2, and is multiplied by the constant β_2. The limit of the product of β_2 and the integral vanishes everywhere except in an infinitesmally small circle about the poles

$$\lim_{\eta_2^2 \to 1} \beta_2 \pi(u, \eta_2^2, \chi_2) \to 0, \qquad u \neq \pm \frac{\pi}{2} .$$

For $u = \pm \frac{\pi}{2}$, the product is indeterminate. This singularity is a reflection of the fact that the celestial longitude is indeterminate in the neighborhood of the celestial poles. If the position in the neighborhood of the poles is not required, then no difficulty arises for polar orbits.

5. Periods of the Motion

When the coordinate ξ goes through one vibration, the argument v increases by 2π, and the time increases by one period. Similarly, as each argument u, φ increases by 2π, the time increases by an amount equal to the periods in these coordinates. Define the periods

T_1 to be the time required for v to change by 2π ,
T_2 to be the time required for u to change by 2π ,
T_3 to be the time required for φ to change by 2π .

The frequencies of the motion are given by

$$\nu_i = \frac{1}{T_i} , \qquad (i = 1, 2, 3) . \tag{31}$$

Define the action variable J_i by the integrals

$$\begin{aligned} J_1 &= \oint P_\xi \, d\xi , \\ J_2 &= \oint P_\eta \, d\eta , \\ J_3 &= \oint P_\varphi \, d\varphi . \end{aligned} \tag{32}$$

Then the frequencies of the motion are given by [Ref. 15, 16]:

(33)
$$\nu_i = \frac{\partial H}{\partial J_i} ,$$

where H is the Hamiltonian. Since the J_i are functions of the β_i $(i = 1, 2, 3)$ only, then the variations of the J_i may be written

(34)
$$dJ_i = \sum_{j=1}^{3} \frac{\partial J_i}{\partial \beta_j} d\beta_j ,$$

where the β_i only are regarded as variables. These equations have a unique inverse, so that

(35)
$$d\beta_j = \sum_{i=1}^{3} \frac{\partial \beta_j}{\partial J_j} dJ_i$$

can be found. Noting the β_1 is the total energy, i.e., $\beta_1 = H$, then only the first equation of the set is required to find the frequencies, that is

(36)
$$d\beta_1 = \sum_{i=1}^{\infty} \nu_i \, dJ_i .$$

Thus the periods can be determined from the kinetic equations of motion by means of these equations and simple algebraic manipulation. The results are

(37)
$$\begin{aligned} T_1 &= L_1(2\pi) + \frac{N_1(2\pi)}{N_2(2\pi)} L_2(2\pi) , \\ T_2 &= \frac{N_2(2\pi)}{N_1(2\pi)} T_1 , \\ T_3 &= \frac{2\pi \, T_1}{\frac{N_1(2\pi)}{N_2(2\pi)} M_2(2\pi) - M_1(2\pi)} , \end{aligned}$$

where the L_i, M_i, N_i are the integrals defined in (4), and T_1, T_2, T_3 are defined above. Series expansions for the periods are readily obtained, but do not appear to be worthwhile, since each of the quantities appearing is computed during the computation of an ephemeris.

6. The Numerical Inversion of the Kinetic Equations

An approximate analytical inversion of the equations (3) may be performed with the aid of the Jacobian elliptic functions. However, in the formulation given in this report the inversion is performed numerically.

Define the integers n and m by

$$n = \text{G. I.}\left[\frac{t_k - t_o}{T_1}\right] ,$$

(38)

$$m = \text{G. I.}\left[\frac{t_k - t_o}{T_2}\right] ,$$

where G. I. $[\quad]$ is to be interpreted to mean "the greatest integer of," and t_o, t_1, t_2, ..., t_k, ... are arbitrary reference times. Define

(39)

$$\sigma^* = (t_o - t_k) + \alpha_1 + nL_1(2\pi) + mL_2(2\pi) ,$$

$$\Omega^* = -\alpha_2 - nM_1(2\pi) + mM_2(2\pi) ,$$

$$\omega^* = -\alpha_3 + nN_1(2\pi) - mN_2(2\pi) .$$

The equations (3) become

(40)

$$(t - t_k) - \sigma^* = L_1(v) + L_2(u) ,$$

$$\varphi - \Omega^* = -M_1(v) + M_2(u) ,$$

$$-\omega^* = N_1(v) - N_2(u) .$$

The first and last of these equations are inverted numerically to obtain u and v at a given time t. The Jacobian of these equations is given by

(41) $$J = \frac{-1}{\sqrt{-2\beta_1}}\frac{1}{\sqrt{1-\chi_1^2\sin^2 v}}\frac{1}{\sqrt{1-\chi_2^2\sin^2 u}}\left\{\frac{\beta_3\rho^2(1+\ell\cos v)^2}{\eta_1 c(1+s_1\cos v)^2} + Q\eta_1 c\chi_2^2\sin^2 u\right\} .$$

The Jacobian is different from zero everywhere. This is a sufficient condition to insure the existence of a unique inverse to these equations.

In these equations, if the reference time t_k is fixed, the "elements" σ^*, Ω^*, and ω^* remain constant, and the arguments u and v increase with time. To avoid overflow of the computer registers, the reference time may be changed from t_k to t_{k+1} at invervals. The elements σ^*, Ω^*, ω^* will be changed at each change of reference time. If the reference time adjustment is made after each revolution of u and v, the elements σ^*, Ω^*, ω^* will, to a good approximation, be equal to the time of perigee passage, the right ascension of the node, and the argument of perigee in perturbed Keplerian motion.

At time $t = t_o$, from (38), (39) and (40),

$$\begin{aligned} \sigma^* &= \alpha_1 , \\ \Omega^* &= -\alpha_2 , \\ \omega^* &= -\alpha_3 . \end{aligned} \tag{42}$$

Therefore, the canonical constants α_1, α_2, α_3 are not independent of the reference time, so that a specification of the canonical constants without a reference time is meaningless.

7. Orbital Elements

The choice of a set of "elements of the orbit" is not arbitrary. A choice dictated by the operational demands of a tracking complex, for example, might be entirely different from a choice based upon other considerations. A number of factors enter, the more pertinent of which appear to be

(a) the information content,

(b) computational convenience,

(c) physical and geometrical significance,

(d) general usage,

all of which must be placed in proper relationship to the controlling criteria of simplicity, overall objectives, and so on. More esoteric considerations may well be appropriate in an analytical study, but hardly appear contributory to a computer-oriented presentation.

To illustrate the variety of choices available, consider the semi-major axis **a**. In Keplerian motion, the semi-major axis is simply related to the total energy, therefore, by analogy, one definition might be

$$a \equiv -\frac{1}{2\beta_1} .$$

In Keplerian motion the semi-major axis is also the mean sum of apogee and perigee distance; hence another definition

$$a \equiv \frac{\xi_1 + \xi_2}{2} ,$$

where ξ_1 and ξ_2 are the maximum and minimum of the coordinate ξ (i.e., the roots of $R(\xi) = 0$). Similarly, another choice is

$$a \equiv \frac{r_1 + r_2}{2} ,$$

where r_1, and r_2 are radial distances. Here two choices are possible:

(a) using the relationship between the polar coordinate and the coordinates

$$r_{1,2} = \sqrt{\xi_{1,2}^2 + c^2(1 - \eta^2)}$$

then (a) r_1, r_2 may be defined as those values at maximum and minimum of ξ, where the value of η is quite simply determined, or (b) r_1, r_2 may be taken as the true maximum and minimum or r, where the values of ξ, η are not so easily determined.

Since the semi-major axis is related to the anomalistic period in Keplerian motion, this also yields a variety of choices.

Each of these definitions yields nearly equal numerical values for the "semi-major axis," and one or another may be adopted with impunity.*

*If each of the above definitions of a is expressed in terms of the period of a Keplerian orbit, the range of the values is about four seconds of time for the satellite Explorer IV.

If consistency is to be maintained, the choice will affect the definition of the eccentricity. Certainly the numerical value of the eccentricity will be affected, and the time variations of both elements under perturbing forces will be affected. It is not known whether this last is an important consideration in maintaining precision.

The set of elements which appear best suited to the objectives of this report are the osculating Keplerian elements. They have the advantages of simplicity, of easy physical and geometrical interpretation, and of antiquity of usage. They contain the maximum amount of information about the motion which can be given independently of the physical model. By obtaining the "best fit" of the path to observations during successive time intervals, the "time variation" of the osculating elements may be obtained. These time variations are of greatest interest in the study of physical models.

8. Computation of an Ephemeris

The principal steps in the computation of an ephemeris are

(a) the computation of the canonical elements from initial conditions at some starting time,

(b) the computation of ξ, η, φ at some other specified time.

The computation of the canonical constants is conveniently made using geocentric rectangular coordinates. The equations for ξ_o, η_o, φ_o at the starting time t_o are

(43)
$$\xi_o = +\sqrt{\frac{r^2 - c^2}{2}\left\{1 + \sqrt{1 + \frac{4z^2c^2}{(r^2 - c^2)^2}}\right\}} ,$$

$$|\eta_o| = \frac{1}{c^2}\sqrt{\frac{r^2 - c^2}{2}\left\{1 - \sqrt{1 + \frac{4z^2c^2}{(r^2 - c^2}}\right\}} ,$$

$$\varphi_o = \tan^{-1}\frac{y}{x} ,$$

where x, y, z are the geocentric rectangular coordinates,

(44)
$$r^2 = x^2 + y^2 + z^2 ,$$

and c is the small constant

$$c^2 = \frac{2}{3} J \cong 1.082 \times 10^{-3} .$$

The coordinate ξ_o is always positive; the sign of the coordinate η_o is taken to be the same as the sign of z. The usual quadrant checks are used in the computation of φ_o. In practice, the radical in the expression for $|\eta_o|$ should be replaced by a power series in c^2 to avoid loss of significance in single precision computations.

The generalized momenta are computed from

$$(45) \qquad \begin{aligned} p_\xi &= \frac{\eta c^2 \dot{z} + \xi r \dot{r}}{\xi^2 + c^2} , \\ p_\eta &= \frac{\xi \dot{z} - \eta r \dot{r}}{1 - \eta^2} , \\ p_\varphi &= x\dot{y} - y\dot{x} , \end{aligned}$$

where the dot indicates the time derivative, and

$$(46) \qquad r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z} .$$

The canonical constants β_1, β_2, β_3 are obtained from

$$(47) \qquad \begin{aligned} \beta_1 &= \frac{1}{2} V^2 - \frac{\xi}{\xi^2 + \eta^2 c^2} , \\ \beta_2 &= p_\varphi , \\ \beta_3 &= \left\{(1 - \eta^2) p_\eta^2 + \frac{\beta_2^2}{1 - \eta^2} - 2\beta_1 \eta^2 c^2\right\}^{\frac{1}{2}} , \end{aligned}$$

where

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 .$$

(Note that p_ξ is not used when the β_i's are expressed in this manner.)

The computation of the canonical constant α_1, α_2, α_3 proceeds as follows:

(a) the roots ξ_1, ξ_2,

(b) the intermediate constants j, k, A, B^2, C, s_1, ℓ, ρ, χ_1^2, Q, q, s^2, I_1, R_1, R_2, φ_1, φ_2,

(c) the argument

$$v = \cos^{-1}\left|\frac{\rho - \xi}{s_1\xi - \rho^2}\right|,$$

with quadrant check,

Sign of p_ξ:	+	+	-	-
Sign of cos v:	+	-	-	+
Quadrant of V:	I	II	III	IV

(d) the integrals $L_1(v)$, $M_1(v)$, $N_1(v)$,

(e) the roots η_1, η_2,

(f) the modulus χ_2^2,

(g) argument

$$u = \sin^{-1}\left|\frac{\eta}{\eta_2}\right|,$$

with quadrant check,

Sign of p_η:	+	-	-	+
Sign of η:	+	+	-	-
Quadrant of u:	I	II	III	IV

(h) the integrals $L_2(u)$, $M_2(u)$, $N_2(u)$,

(i) the constants α_1, α_2, α_3.

The determination of the coordinates at some other specified time t requires the numerical inversion of the equations

$$(t - t_o) - \alpha_1 = L_1(v) + L_2(u) ,$$
$$\alpha_3 = N_1(v) - N_2(u) .$$

This may be accomplished in a number of ways. A Newton-Raphson procedure works well in practice. Define v_j, u_j to be the j^{th} trial values of v, u, corresponding to values α_{1j}, α_{3j}. Then the $(j + 1)$ trial value is given by

$$v_{j+1} = v_j + \left(\frac{\partial v}{\partial \alpha_1}\right)(\alpha_1 - \alpha_{1j}) + \left(\frac{\partial v}{\partial \alpha_3}\right)(\alpha_3 - \alpha_{3j}) ,$$

(49)

$$u_{j+1} = u_j + \left(\frac{\partial v}{\partial \alpha_1}\right)(\alpha_1 - \alpha_{1j}) + \left(\frac{\partial v}{\partial \alpha_3}\right)(\alpha_3 - \alpha_{3j}) ,$$

where the partial derivatives are given by

$$\frac{\partial v}{\partial \alpha_1} = - \frac{\sqrt{1 - \chi_1^2 \sin^2 v}}{Q(\xi^2 + \eta^2 c^2)} , \qquad \frac{\partial v}{\partial \alpha_3} = - \frac{\eta^2 c^2 \sqrt{1 - \chi_1^2 \sin^2 v}}{\beta_3 Q(\xi^2 + \eta^2 c^2)} ,$$

(50)

$$\frac{\partial u}{\partial \alpha_1} = + \frac{\sqrt{1 - \chi_2^2 \sin^2 u}}{(\xi^2 + \eta^2 c^2)} , \qquad \frac{\partial u}{\partial \alpha_3} = + \frac{\xi^2 \sqrt{1 - \chi_2^2 \sin^2 u}}{\sqrt{-2\beta_1 c^2} \quad \eta_1(\xi^2 + \eta^2 c^2)} ,$$

where ξ, η are determined from the j^{th} trial values of u, v .

With these values of u, v, the coordinates ξ, η, φ, p_ξ, p_η, p_φ may be obtained. With the transformation of these coordinates to the appropriate reference frame (e.g., range, right ascension and declination, or azimuth, elevation and height above geoid, etc.) the computation is complete.

9. Approximate Expressions for the Kinetic Equations

Values of the small quantities j, k, ℓ, χ_1^2, χ_2^2 are at most $O(c^2)$. For example, for an earth satellite in an orbit similar to the orbit of Explorer IV near the date of launch,*

$$
\begin{aligned}
j &\cong -2.2 \times 10^{-5} \\
k &\cong 6.5 \times 10^{-4} \\
\ell &\cong -3.5 \times 10^{-5} \\
\chi_1^2 &\cong 8 \times 10^{-6} \\
\chi_2^2 &\cong 5 \times 10^{-4}
\end{aligned}
$$

Simplified expressions for the kinetic equations are obtained by ignoring squares of these small quantities in the nonsecular terms.

It is convenient to express the results in terms of an angle variable analogous to the eccentric anomaly in Keplerian motion. Make the transformation on the angle v

(51)
$$\cos v = \frac{\cos E^* - s_1}{1 - s_1 \cos E^*},$$
$$\sin v \sqrt{1 - \chi_1^2 \sin^2 v} = \frac{\sqrt{1 - s_1^2}\, \sin E^* \sqrt{1 - \chi_1^2 \sin^2 E^*}}{1 - s_1 \cos E^*}.$$

Substituting (51) into the homographic transformation (14) yields

(52)
$$\xi = a^*[1 - \frac{(s_1 - \ell)}{1 - \ell s_1} \cos E^*],$$

where a^* is defined by

(53)
$$a^* = \frac{\xi_1 + \xi_2}{2}.$$

---

*The units are such that $(G^2M)/R^2 = 1$, where G^2 is the universal constant of gravitation, M is a representative mass of the earth, and R is a representative earth radius.

The integral $L_1(\xi)$ defined by equation (4) becomes

$$(54) \qquad L_1(\xi) = a^{*2} Q \sqrt{1 - s_1^2} \left\{ F(E^*, \chi_1) - \left[2\left(\frac{\xi_1 - \xi_2}{\xi_1 + \xi_2}\right) - s_1 \right] C_1(E^*) + \frac{\ell^2 (1 - s_1^2)^2}{(1 - \ell s_1)^2} \int \frac{\cos^2 E^* \, dE^*}{(1 - s_1 \cos E^*) \sqrt{1 - \chi_1^2 \sin^2 E^*}} \right\},$$

where $C_1(E^*)$ is defined in (20).

The last term in (54) contains a secular part. However, for arcs of about 100 revolutions or less, and for $s_1^2 < 1$, this term is less than 10^{-6} in magnitude. Omitting this term from (54),

$$(55) \qquad L_1(\xi) = \frac{1}{n^*} [F(E^*, \chi_1) - e^* C_1(E^*)] + O(\ell^2) ,$$

where n^* and e^* are defined by

$$(56) \qquad n^* = (a^{*2} Q \sqrt{1 - s_1^2})^{-1} , \qquad e^* = 2\left(\frac{\xi_1 - \xi_2}{\xi_1 + \xi_2}\right) - s_1 .$$

Omitting terms in χ_1^4 from (55) yields

$$(57) \quad L_1(\xi) = \frac{1}{n^*} \left\{ E^* - e^* \sin E^* + \frac{\chi_1^2}{4} \left[E^* - \sin E^* \cos E^* + \frac{2e^*}{3} \sin^3 E^* \right] \right\} + O(\chi_1^4, \ell^2) .$$

If justification exists for retaining higher order terms in χ_1^2 in the secular parts of (54), while omitting the term in ℓ^2, this can be done by modifying (57) to include the complete elliptic integral of the first kind evaluated to order χ_1^4, i.e.,

$$\frac{2}{\pi} K_1(\chi_1) = 1 + \frac{\chi_1^2}{4} + \frac{9\chi_1^4}{64} + O(\chi_1^6) .$$

Thus (57) becomes

$$(58)\quad L_1(\xi) \cong \frac{1}{n^*}\left\{\left(\frac{2K(\chi_1)}{\pi}\right) E^* - e^* \sin E^* - \frac{\chi_1^2}{4} \sin E^*\left(\cos E^* - \frac{2e^*}{3} \sin^2 E^*\right) + \cdots\right\}$$

The integral $L_2(\eta)$ defined in (4) can be reduced by subtracting the elliptic integral of the second kind from the elliptic integral of the first kind, term by term. The result,

$$(59)\quad L_2(\eta) \cong \left(\frac{\eta_1 c}{\sqrt{-2\beta_1}}\right)\left(\frac{\chi_2^2}{2}\right)\left\{\left(1 + \frac{3}{8}\chi_2^2\right)u - \sin u \cos u + \cdots\right\} .$$

Terms in χ_2^4 have been retained in the secular part of (59).

The integral $M_1(\xi)$ defined in (4) is conveniently reduced from the form

$$(60)\quad M_1(\xi) = \frac{\beta_2 c^2 Q}{\rho^2 + c^2}\left\{ C_o(v) + \rho(s_1 - \ell)\left[\frac{2\rho C_1(v)}{\rho^2 + c^2}\right.\right.$$

$$- \left\{\frac{(3\rho^2 - c^2)\rho\ell - \rho s_1(\rho^2 - 3c^2)}{(\rho^2 + c^2)^2}\right\} C_2(v)$$

$$+ \left\{\frac{4\rho(\rho^2 - c^2)(\rho^2\ell^2 - c^2 s_1^2) - \rho\ell s_1[(\rho^2 - c^2)^2 - 4\rho^2 c^2]}{(\rho^2 + c^2)^3}\right\} C_3(v)$$

$$- \left\{\frac{(\rho^3\ell^3 - \rho\ell c^2 s_1^2)(5\rho^4 - 10\rho^2 c^2 - c^4) - (\rho^2\ell^2 s_1 + c^2 s_1^3)(\rho^5 - 10\rho^3 c^2 + 5\rho c^4)}{(\rho^2 + c^2)^4}\right\} C_4(v)$$

$$\left.\left. + \cdots \right]\right\} ,$$

where the $C_m(v)$ have been defined in (20).

Omitting terms in c^4, $c^2\ell$, ℓ^2, and $c^2\chi_2^2$, then (61) reduces to

$$(62)\quad M_1(\xi) \cong \frac{\beta_2 c^2 Q}{\rho^2}\left\{ v\left(1 + \frac{s_1}{2\rho} + \cdots\right) + 2s_1 \sin v + \cdots \right\} .$$

The integral $M_2(\eta)$ to terms of order χ_2^2 is given by

$$(63)\quad M_2(\eta) \cong \left\{\frac{\beta_2}{\eta_1 c\sqrt{-2\beta_1}}\right\}\left\{\frac{1}{\sqrt{1 - \eta_2^2}} \tan^{-1}\left(\sqrt{1-\eta_2^2} \tan u\right)\left(1 + \frac{\chi_2^2}{2\eta_2^2} + \cdots\right) - \frac{\chi_2^2}{2\eta_2^2} u + \cdots \right\} .$$

(64) $$N_1(\xi) \cong Q\,\beta_3\{[\frac{2K(\chi_1)}{\pi}]\}v - \frac{\chi_1^2}{4}\sin v\cos v + \cdots\} .$$

The integral $N_2(\eta)$ is given approximately by

(65) $$N_2(\eta) \cong \frac{\eta_1 c\beta_3}{\sqrt{-2\beta_1}}\{[\frac{2K(\chi_2)}{\pi}]u - \frac{\chi_2^2}{4}\sin u\cos u\} .$$

The transformation between E^* and v given in (51) may be conveniently written in the form

(64) $$\tan v = \frac{\sqrt{1-s_1^2}\sqrt{1-\chi_1^2\sin^2 E^*}\sin E^*\,(1-s_1\cos E^*)}{\sqrt{(1+s_1^2-2s_1\cos E^*)-[s_1^2(1-\chi_1^2)+\chi_1^2]\sin^2 E^*}\,(\cos E^*-s_1)} .$$

To the order χ_1^2, this becomes

(67) $$\tan v = \sqrt{1-s_1^2}\left(\frac{\sin E^*}{\cos E^*-s_1}\right)\left(1+\frac{\chi_1^2 s_1\sin^2 E^*\cos E^*(2-s_1\cos E^*)}{2(1-s_1\cos E^*)^2}\right) .$$

Thus for small eccentricities (i.e., $s_1 \ll 1$), the well-known "half-angle formula" applies, that is

(68) $$\tan\frac{v}{2} \cong \sqrt{\frac{1+s_1}{1-s_1}}\tan\frac{E^*}{2} .$$

Approximate values of the constants are given by

(69) $$s_1 = \left(\frac{\xi_1-\xi_2}{\xi_1+\xi_2}\right)\left(1-\frac{k}{2\xi_1\xi_2}\right) + O(k^2) ,$$

(70) $$e^* = \left(\frac{\xi_1-\xi_2}{\xi_1+\xi_2}\right)\left(1+\frac{4j}{\xi_1+\xi_2}+\frac{3k}{2\xi_1\xi_2}\right) + O(jk) ,$$

(71) $$\sqrt{-2\beta_1}\,Q = \frac{1}{\sqrt{\xi_1\xi_2}}\left[1+\left(\frac{\xi_1+\xi_2}{2\xi_1\xi_2}\right)\left(j+\frac{k(\xi_1-\xi_2)}{2\xi_1\xi_2}\right)\right] + O(k^2) ,$$

where j,k are defined by (15).

10. Coefficients for Differential Improvement of the Orbit

The differences between the observed and computed positions of the object are to be used to obtain improvements in the constants of the motion. The linear equations which are to be solved are of the form

$$\{\Delta(O - c)\} = \{T\} \begin{Bmatrix} d\alpha_1 \\ d\alpha_2 \\ \vdots \\ d\beta_3 \end{Bmatrix} \tag{72}$$

where $\{\Delta(O - c)\}$ is a column matrix whose elements are the differences between the observed and computed positions of the object, $\{T\}$ is a matrix whose elements are partial derivatives of the coordinates of the object with respect to the constants of the motion, and $d\alpha_1$, $d\alpha_2$, ... $d\beta_3$ are (small) corrections to the constants of the motion.

Throughout the following, it is assumed that a solution to (72) can be obtained.

The purpose of this section is to present the elements of the matrix $\{T\}$. While in principle these elements are easy to obtain, the actual expressions are quite lengthy. This leads to "bookkeeping" problems. To simplify the presentation, the results are recorded as a number of matrices, each of which contains the results of an intermediate step. The final equations may be obtained by multiplying these matrices; if, indeed, this is done, it will be found that only a few minor cancellations and simplifying combinations result, so that the matrix representation is reasonably economical. Some loss in computational efficiency no doubt results if the computer coding is carried out directly from the matrix representations, but in view of the other advantages, the loss in efficiency is not believed to be important.

The residuals may be obtained by subtracting the computed values from observed values. However, an approximate, more efficient procedure is provided by the equations

$$\text{(73)} \quad \{\Delta(O - c)\} = \begin{Bmatrix} \rho \cos\beta \\ \rho \, d\beta \\ d\rho \end{Bmatrix} = \begin{Bmatrix} (x-X)\sin\theta - (y-Y)\cos\theta \\ [(x-X)^2 + (y-Y)^2]^{\frac{1}{2}}\sin\beta - (z-Z)\cos\beta \\ \rho_{obs} - \rho_{comp} \end{Bmatrix}$$

where

x, y, z are computed geocentric rectangular coordinates of the object at the time of observation;

X, Y, Z are computed geocentric rectangular coordinates of the observer at the time of observation;

θ, β are observed right ascension and declination;

ρ is the slant range from observer to object.

If the slant range is not observed, the last row in the column vector is omitted.

The residuals must be transformed into oblate spheroidal coordinates. Define the rotation matrix $\{T_1\}$ by

$$\text{(74)} \quad \{\Delta(O - c)\} = \begin{Bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\beta & -\sin\theta\sin\beta & \cos\beta \\ \cos\theta\cos\beta & \sin\theta\cos\beta & \sin\beta \end{Bmatrix} \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix}$$

Define the rotation matrix $\{T_2\}$ by

$$\text{(75)} \quad \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} \dfrac{x\xi}{\xi^2 + c^2} & \dfrac{-x\eta}{1 - \eta^2} & -y \\ \dfrac{y\xi}{\xi^2 + c^2} & \dfrac{-y\eta}{1 - \eta^2} & x \\ \eta & \xi & 0 \end{Bmatrix} \begin{Bmatrix} d\xi \\ d\eta \\ d\varphi \end{Bmatrix}$$

Thus, the residuals may be written

$$(76)\qquad \{\Delta(O - C)\} = \{T_1\}\{T_2\}\begin{Bmatrix} d\xi \\ d\eta \\ d\varphi \end{Bmatrix}$$

where $\{T_1\}$, $\{T_2\}$ are defined by (74) and (75).

The differentials in the oblate spheroidal coordinates are related to differentials in the constants of the motion through the kinetic equations. Write the kinetic equations in the form

$$(77)\qquad \begin{aligned} (t - t_o) - \alpha_1 &= L_1(\beta_i, \xi) + L_2(\beta_i, \eta) , \\ \alpha_2 &= -M_1(\beta_i, \xi) + M_2(\beta_i, \eta) - \varphi, \\ \alpha_3 &= N_1(\beta_i, \xi) - N_2(\beta_i, \eta) . \end{aligned}$$

Differentiate both sides with respect to the coordinates, as they appear explicitly. The resulting linear equations have a unique solution:

$$(78)\qquad \begin{Bmatrix} d\xi \\ d\eta \\ d\varphi \end{Bmatrix} = \{T_3\}\left[\{I\}\begin{Bmatrix} -d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{Bmatrix} - \{I\}\begin{Bmatrix} -d\alpha_1' \\ d\alpha_2' \\ d\alpha_3' \end{Bmatrix}\right],$$

where $\{T_3\}$ is the matrix defined by

$$(79)\qquad \{T_3\} = \frac{1}{\xi^2 + \eta^2 c^2}\begin{Bmatrix} (\xi^2 + c^2)p_\xi & 0 & \dfrac{\eta^2 c^2}{\beta_3}(\xi^2 + c^2)p_\xi \\ (1 - \eta^2)p_\eta & 0 & -\xi^2(1 - \eta^2)p_\eta \\ \dfrac{-\beta_2 c^2}{(\xi^2 + c^2)} & -(\xi^2 + \eta^2 c^2) & \dfrac{-\beta_2}{\beta_3}\left[\dfrac{\xi^2}{1 - \eta^2} + \dfrac{c^4\eta^2}{\xi^2 + c^2}\right] \end{Bmatrix}.$$

$\{I\}$ is the unit matrix, defined by

$$\{I\} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} ,$$

and $d\alpha_1'$, $d\alpha_2'$, $d\alpha_3'$ are total differentials in α_1, α_2, α_3 regarded as functions of β_1, β_2, β_3 only, that is,

(80) $$d\alpha_j' = \sum_{i=1}^{3} \frac{\partial \alpha_j}{\partial \beta_i} d\beta_i , \qquad (j = 1,2,3) .$$

The $d\alpha_1'$, $d\alpha_2'$, $d\alpha_3'$ are to be expressed in terms of the $d\beta_1$, $d\beta_2$, $d\beta_3$. Using equations (58), (59), (62), (63), (64) and (65), the kinetic equations (77) may be differentiated with respect to the constants which appear explicitly. The result is the 3×12 matrix $\{D(m, n)\}$,

(81) $$\begin{Bmatrix} d\alpha_1' \\ d\alpha_2' \\ d\alpha_3' \end{Bmatrix} = \{D(m, n)\} \begin{Bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \\ d(\eta_1 c) \\ d\eta_2 \\ da^* \\ dQ' \\ ds_1 \\ de^* \\ d\chi_1^2 \\ d\chi_2^2 \\ d\rho \end{Bmatrix} ,$$

where $Q' = (Q\sqrt{-2\beta_1'})$.

The first column of $\{D(m, n)\}$ is made up of elements which are the (explicit) derivatives of the kinetic equations with respect to β_1, the second column is made up of the derivatives with respect to β_2, etc. Listing by column and by element:

<u>1st column (β_1):</u>

$$D(1,1) = \frac{t - t_o - \alpha_1}{-2\beta_1} \qquad D(2,1) = \frac{\alpha_2 + \varphi}{-2\beta_1} \qquad D(3,1) = \frac{\alpha_2}{-2\beta_1}$$

<u>2nd column (β_2):</u>

$$D(1,2) = 0 \qquad D(2,2) = \frac{\alpha_2 + \varphi}{\beta_2} \qquad D(3,2) = 0$$

<u>3rd column (β_3):</u>

$$D(1,3) = 0 \qquad D(2,3) = 0 \qquad D(3,3) = \frac{\alpha_2}{\beta_3}$$

<u>4th column ($\eta_1 c$).</u>

$$D(1,4) = \frac{L_2(\eta)}{\eta_1 c} \qquad D(2,4) = \frac{-M_2(\eta)}{\eta_1 c} \qquad D(3,4) = \frac{N_2(\eta)}{\eta_1 c}$$

<u>5th column (η_2):</u>

$$D(1,5) = 0 \qquad D(2,5) = \left\{\frac{\beta_2}{\eta_1 c\sqrt{-2\beta_1}}\right\}\left\{\left[\frac{\tan^{-1}(\sqrt{1-\eta_2^2}\ \tan u)}{(1-\eta^2)^{3\ 2}}\ [\eta_2 + \frac{\chi_2^2}{2\eta_2^2}\ (3\eta_2^2-2)]\right] + \frac{\chi_2^2}{2\eta_2^2}\ u\right\}$$

$$D(3,5) = 0$$

<u>6th column (a^*):</u>

$$D(1,6) = \frac{2L_1(\xi)}{a^*} \qquad D(2,6) = 0 \qquad D(3,6) = 0$$

<u>7th column (Q'):</u>

$$D(1,7) = \frac{L_1(\xi)}{Q'} \qquad D(2,7) = \frac{-M_1(\xi)}{Q'} \qquad D(3,5) = \frac{N_1(\xi)}{Q'}$$

8th column (s_1):

$$D(1,8) = \frac{-s_1 L_1(\xi)}{1 - s_1^2} \qquad D(2,8) = \frac{\beta_2 Q' c^2}{2\sqrt{-2\beta_1}\,\rho^3}[v + 4\rho \sin v + \cdots] \qquad D(3,8) = 0$$

9th column (e^*):

$$D(1,9) = \frac{-C_1(E^*)}{n^*} \cong \frac{-\sin E^*}{n^*}[1 + \frac{\chi_1^2}{6} \sin^2 E^* + \cdots] \qquad D(2,9) = 0 \qquad D(3,9) = 0$$

10th column (χ_1^2):

$$D(1,10) = \frac{1}{4n^*}[(1 + \frac{9}{8}\chi_1^2)E^* - \sin E^*(\cos E^* - \frac{2e^*}{3}\sin^2 E^*) + \cdots] \qquad D(2,10) = 0$$

$$D(3,10) = \frac{Q'\beta_3}{4\sqrt{-2\beta_1}}[(1 + \frac{9}{8}\chi_1^2)v - \sin v \cos v + \cdots]$$

11th column (χ_2^2):

$$D(1,11) = \frac{\eta_1 c}{2\sqrt{-2\beta_1}}[(1 + \frac{3}{4}\chi_2^2)u - \sin u \cos u + \cdots]$$

$$D(2,11) = \frac{\beta_2}{2\eta_1 c\sqrt{-2\beta_1}\,\eta_2^2}\left[\frac{\tan^{-1}(\sqrt{1-\eta_2^2}\tan u)}{\sqrt{1-\eta_2^2}} - u\right]$$

$$D(3,11) = \frac{\beta_3}{\eta_1 c\sqrt{-2\beta_1}}[(1 + \frac{9}{8}\chi_2^2)u - \sin u \cos u + \cdots]$$

12th column (ρ):

$$D(1,12) = 0 \qquad D(2,12) = \frac{2M_1(\xi)}{\rho} + \frac{s_1 c^2 \beta_2 Q' v}{2\sqrt{-2\beta_1}\,\rho^3} + \cdots \qquad D(3,12) = 0$$

The constants appearing explicitly in the kinetic equations are in turn explicit functions of ξ_1, ξ_2, j, k, $\cdots$.

Define the matrix $\{D'(m, n)\}$ by

(82)
$$\begin{Bmatrix} \begin{bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \\ d(\eta_1 c) \\ d\eta_2 \\ da^* \\ dQ' \\ ds_1 \\ de^* \\ d\chi_1^2 \\ d\chi_2^2 \\ d\rho \end{bmatrix} \end{Bmatrix} = \{D'(m, n)\} \begin{Bmatrix} \begin{bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \\ d(\eta_1 c) \\ d\eta_2 \\ d\xi_1 \\ d\xi_2 \\ dj \\ dk \end{bmatrix} \end{Bmatrix} .$$

The matrix $\{D'(m, n)\}$ is conveniently partitioned into 3×3 submatrices. Define I to be the 3×3 unit matrix, and 0 to be a 3×3 null matrix. Then write

(83)
$$\{D'(m, n)\} = \begin{Bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & d'(2,2) & d'(2,3) \\ 0 & d'(3,2) & d'(3,3) \\ 0 & d'(4,2) & d'(4,3) \end{bmatrix} \end{Bmatrix} .$$

Define the functions

(84)
$$H_1^2 = \xi_1^2 + 2j\xi_1 + k ,$$
$$H_2^2 = \xi_2^2 + 2j\xi_2 + k ,$$
$$G_1 = \left(\frac{\xi_1 - \xi_2}{2}\right)^2 Q'^2 + \frac{2s_1}{(1 + s_1^2)^2} ,$$
$$G_2 = \left(\frac{\xi_1 - \xi_2}{2}\right)^2 Q'^2 - \frac{2s_1}{(1 + s_1^2)^2} .$$

The submatrices are given by

(85) $$d'(2,2) = \left\{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{matrix}\right\},$$

(86) $$d'(2,3) = \left\{\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{matrix}\right\},$$

(87) $$d'(3,2) \left\{\begin{matrix} 0 & 0 & -\frac{Q'}{2}\left(\frac{\xi_1 + j}{H_1^2}\right) \\ 0 & 0 & \left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_1 + j}{H_1^2}\right) \\ 0 & 0 & \frac{\xi_2}{a*^2} - \left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_1 + j}{H_1^2}\right) \end{matrix}\right\}$$

(88) $$d'(3,3) = \left\{\begin{matrix} -\frac{Q'}{2}\left(\frac{\xi_2 + j}{H_2^2}\right) & -\frac{Q'}{2}\left(\frac{\xi_1}{H_1^2} + \frac{\xi_2}{H_2^2}\right) & -\frac{Q'}{4}\left(\frac{1}{H_1^2} + \frac{1}{H_2^2}\right) \\ -\left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_2 + j}{H_2^2}\right) & \left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_1}{H_1^2} - \frac{\xi_2}{H_2^2}\right) & \left(\frac{1 - s_1^2}{4}\right)\left(\frac{1}{H_1^2} - \frac{1}{H_2^2}\right) \\ -\frac{\xi_1}{a*^2} + \left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_2 + j}{H_2^2}\right) & -\left(\frac{1 - s_1^2}{2}\right)\left(\frac{\xi_1}{H_1^2} - \frac{\xi_2}{H_2^2}\right) & -\left(\frac{1 - s_1^2}{4}\right)\left(\frac{1}{H_1^2} - \frac{1}{H_2^2}\right) \end{matrix}\right\}$$

(89) $$d'(4,2) = \left\{\begin{matrix} 0 & 0 & \left(\frac{\xi_1 - \xi_2}{2}\right) Q'^2 - G_1 \left(\frac{\xi_1 + j}{H_1^2}\right) \\ -\frac{2\chi_2^2}{\eta_1 c} & -\frac{2\chi_2^2}{\eta_2} & 0 \\ 0 & 0 & \left(\frac{1 - s_1}{2}\right) - \frac{(\xi_1 - \xi_2)(1 - s_1^2)(\xi_1 + j)}{4H_1^2} \end{matrix}\right\}$$

$$(90)\quad d'(4,3) = \begin{Bmatrix} -\frac{(\xi_1-\xi_2)}{2}Q'^2 - G_2\frac{(\xi_2+j)}{H_2^2} & -\frac{G_1\xi_1}{H_1^2} - \frac{G_2\xi_2}{H_2^2} & -\frac{G_1}{2H_1^2} - \frac{G_2}{2H_2^2} \\ 0 & 0 & 0 \\ \left(\frac{1+s_1}{2}\right) + \frac{(\xi_1-\xi_2)(1-s_1^2)(\xi_2+j)}{4H_2^2} & \frac{-(\xi_1-\xi_2)(1-s_1^2)}{4}\left(\frac{\xi_1}{H_1^2}-\frac{\xi_2}{H_2^2}\right) & \frac{-(\xi_1-\xi_2)(1-s_1^2)}{8}\left(\frac{1}{H_1^2}-\frac{1}{H_2^2}\right) \end{Bmatrix}$$

As the last step, define the matrix $\{D''(m, n)\}$ by the transformation

$$(91)\quad \begin{Bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \\ d\eta_1 c \\ d\eta_2 \\ d\xi_1 \\ d\xi_2 \\ dj \\ dk \end{Bmatrix} = \{D''(m, n)\} \begin{Bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \end{Bmatrix}$$

Define the quantities

$$(92)\quad \begin{aligned} d_i &= \frac{1}{\eta_i[2\beta_1 c^2(1 - 2\eta_i^2) - \beta^2]} \\ g_i &= \frac{1}{(\xi_i^2 + c^2)(2\beta_1\xi_i + 1) - \xi_i\beta_2^2 c^2} \end{aligned}$$

The elements of $\{D''(m,n)\}$ are given by:

$$
(93)\quad D''(m,n) = \left\{ \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
- c^3\eta_1^2(1-\eta_1^2)d_1 & c\beta_2 d_1 & c\beta_3(1-\eta_1^2)d_1 \\
- c^2\eta_2^2(1-\eta_2^2)d_2 & \beta_2 d_2 & \beta_3(1-\eta_2^2)d_2 \\
- \xi_1^2(\xi_1^2+c^2)^2 g_1 & - \beta_2 c^2(\xi_1^2+c^2)g_1 & \beta_3(\xi_1^2+c^2)^2\ g_1 \\
- \xi_2^2(\xi_2^2+c^2)^2 g_2 & - \beta_2 c^2(\xi_2^2+c^2)g_2 & \beta_3(\xi_2^2+c^2)^2\ g_2 \\
\begin{array}{l} \dfrac{-1}{2\beta_1^2} + \dfrac{1}{2}\Big[\xi_1^2(\xi_1^2+c^2)^2 g_1 \\ \quad + \xi_2^2(\xi_2^2+c^2)^2\ g_2\Big] \end{array} & \begin{array}{l} - \dfrac{c^2\beta_2}{2}\Big[(\xi_2^2+c^2)g_2 \\ \quad + (\xi_2^2+c^2)g_2\Big] \end{array} & \begin{array}{l} \dfrac{\beta_3}{2}\Big[(\xi_1^2+c^2)^2\ g_1 \\ \quad + (\xi_2^2+c^2)^2 g_2\Big] \end{array} \\
\begin{array}{l} - k\Big[\dfrac{1}{\beta_1} + \xi_1(\xi_1^2+c^2)^2 g_1 \\ \quad + \xi_2(\xi_2^2+c^2)^2 g_2\Big] \end{array} & \begin{array}{l} \beta_2 c^2\Big\{\dfrac{1}{2\beta_1\xi_1\xi_2} - k\Big[\dfrac{(\xi_1^2+c^2)g_1}{\xi_1} \\ \quad + \dfrac{(\xi_2^2+c^2)g_2}{\xi_2}\Big]\Big\} \end{array} & \begin{array}{l} \beta_3\Big\{\dfrac{-c^2}{\beta_1\xi_1\xi_2} - k\Big[\dfrac{(\xi_1^2+c^2)^2 g_1}{\xi_1} \\ \quad + \dfrac{(\xi_2^2+c^2)^2 g_2}{\xi_2}\Big]\Big\} \end{array}
\end{array} \right\}
$$

The equation (72) can be written

$$\{\Delta(O-c)\} = \{T_1\}\{T_2\}\{T_3\}\left[\{I\}\begin{Bmatrix} d\alpha_1 \\ d\alpha_2 \\ d\alpha_3 \end{Bmatrix} - \{I\}\{D(m,n)\}\{D'(m,n)\}\{D''(m,n)\}\begin{Bmatrix} d\beta_1 \\ d\beta_2 \\ d\beta_3 \end{Bmatrix}\right. . \tag{94}$$

If the corrections are to be made on elements a, e, i (semi-major axis, eccentricity, inclination) which are defined in terms of the β_i only, an additional matrix multiplication will be required. The elements of the next matrix will, of course, depend upon the definitions. If the elements proposed by Izsak (Ref. 10) defined by

$$a = \frac{\xi_1 + \xi_2}{2}$$

$$e = \frac{\xi_1 + \xi_2}{\xi_1 + \xi_2}$$

$$i = \sin^{-1}|\eta_2|$$

are used, the matrix $\{D''(m,n)\}$ is simplified.

11. Variation of Constants

The differential equations of motion may be written in the form

$$\begin{aligned} \mathring{p}_i + \frac{\partial H}{\partial q_i} &= F_i , \\ \mathring{q}_i - \frac{\partial H}{\partial p_i} &= 0 , \end{aligned} \qquad (i = 1,2,3) , \tag{95}$$

where p_i, q_i are the generalized momenta and coordinates, and F_i is the generalized force acting along the i^{th} coordinate. The equations (3) are to be regarded as solutions to (95) with $F_i = 0$; these solutions may be written in the form

$$C_j = C_j(p_i, q_i, t) , \qquad \begin{matrix} (j = 1,2, \ldots 6) \\ (i = 1,2,3) \end{matrix} , \tag{96}$$

where the C_j are the constants of motion. The solution to (95) may be constructed from the solutions (96) by the so-called "variation of constants" method.

The results may be written

$$\mathring{C}_j = \sum_{i=1}^{3} F_i \frac{\partial H}{\partial p_i} C_j(p_i, q_i, t) . \tag{97}$$

Write the equations for the β_i in the form

$$\begin{aligned} \beta_1 &= H \\ \beta_2 &= p_\varphi \\ \beta_3 &= \sqrt{(1 - \eta^2)p_\eta^2 + \frac{p_\varphi^2}{1 - \eta^2} - 2H\eta c^2} . \end{aligned} \tag{98}$$

Write $\{\nabla_p\}$ for the matrix whose elements are derivatives of the β_i with respect to the momenta,

$$(99)\quad \{\nabla p\} = \left\{\begin{matrix} \dfrac{\xi^2 + c^2}{\xi^2 + \eta^2 c^2}\, p_\xi & \dfrac{1 - \eta^2}{\xi^2 + \eta^2 c^2}\, p_\eta & \dfrac{\beta_2}{(1 - \eta^2)(\xi^2 + c^2)} \\ 0 & 0 & 1 \\ \dfrac{-\eta^2 c^2(\xi^2 + c^2)}{\beta_3(\xi^2 + \eta^2 c^2)}\, p_\xi & \dfrac{\xi^2(1 - \eta^2)}{\beta_3(\xi^2 + \eta^2 c^2)}\, p_\eta & \dfrac{\xi^2 - \eta^2 c^2 + c^2}{\beta_3(1 - \eta^2)(\xi^2 + c^2)} \end{matrix}\right\}.$$

The differential equations for the new "constants of motion" are given by

$$(100)\quad \left\{\begin{matrix} \mathring{\beta}_1 \\ \mathring{\beta}_2 \\ \dot{\beta}_3 \end{matrix}\right\} = \{\nabla_p\} \left\{\begin{matrix} F_\xi \\ F_\eta \\ F_\varphi \end{matrix}\right\}.$$

The differential equations for the α_i as a function of time are obtained in the same manner. The results are

$$(101)\quad \left\{\begin{matrix} -\dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \mathring{\alpha}_3 \end{matrix}\right\} = \{D(m,n)\}\ \{D'(m,n)\}\ \{D''(m,n)\}\ \{\nabla p\} \left\{\begin{matrix} F_\xi \\ F_\eta \\ F_\varphi \end{matrix}\right\},$$

where $\{D(m,n)\}$, $\{D'(m,n)\}$, $\{D''(m,n)\}$, and $\{\nabla(p)\}$ are given by (81), (82), (91), and (99) respectively.

The generalized momenta are given by

$$(102)\quad \left\{\begin{matrix} p_\xi \\ p_\eta \\ p_\varphi \end{matrix}\right\} = \left\{\begin{matrix} -\dfrac{q}{Q} \cdot \dfrac{\sin v\sqrt{1 - \chi_1^2 \sin^2 v}}{(\xi^2 + c^2)\ (1 + s_1 \cos v)} \\ \dfrac{\eta_2}{\sqrt{-2\beta_1 \eta_1 c}} \cdot \dfrac{\cos u\sqrt{1 - \chi_2^2 \sin^2 u}}{(1 - \eta^2)} \\ \beta_2 \end{matrix}\right\}.$$

The generalized forces are related to the forces acting along the x, y, z axes by the transformation

$$
\text{(103)} \quad \begin{Bmatrix} F_\xi \\ F_\eta \\ F_\varphi \end{Bmatrix} = \left(\frac{1}{\xi^2 + \eta^2 c^2}\right) \begin{Bmatrix} x\xi & y\xi & \eta(\xi^2 + c^2) \\ -x\eta & -y\eta & \xi(1 - \eta^2) \\ -y\left(\frac{\xi^2 + \eta^2 c^2}{(\xi^2 + c^2)(1 - \eta^2)}\right) & x\left(\frac{\xi^2 + \eta^2 c^2}{(\xi^2 + c^2)(1 - \eta^2)}\right) & 0 \end{Bmatrix}
$$

If the forces F_i are sufficiently small, the terms of $O(c^2)$ may be put equal to zero in the equations (100), (101), (102), and (103). The results correspond to the equations given by Moulton(Ref. 17) with the substitutions

$$
\begin{aligned}
\beta_1 &\to -\frac{1}{2a}\ , \\
\beta_2 &\to \sqrt{a(1 - e^2)}\ \cos i\ , \\
\beta_3 &\to \sqrt{a(1 - e\ )}\ , \\
\alpha_1 &\to -\delta a^{3/2}\ , \\
\alpha_2 &\to -\Omega\ , \\
\alpha_3 &\to -\omega\ ,
\end{aligned}
$$

where the symbols are those used by Moulton.

BIBLIOGRAPHY

1. Vinti, John P., "New Method of Solution for Unretarded Satellite Orbits," Journal of Research of the National Bureau of Standards - B. Mathematics and Mathematical Physics, Vol. 63B, No. 2, October-December, 1959.

2. Herget, P., and Musen, P., "A Modified Hansen Lunar Theory for Artificial Satellites." The Astronomical Journal 63, No. 10, pp. 430-433. November, 1958.

3. Musen, P., "Application of Hansen's theory to the Motion of an Artificial Satellite in the Gravitational Field of the Earth," Journal of Geophysical Research, Vol. 64, No. 12, pp. 2271-2279. December, 1959.

4. Vienop, E., and Brady, J. L., "The Themis Code: An Astronomical Numerical Integration Program for the IBM 704," Livermore, California, University of California Radiation Laboratory Report 5242. 1958.

5. Baker, R. M. L., et al.,"Efficient Precision Orbit Computation Techniques," American Rocket Society Publication. 1959. 40 pp.

6. King-Hele, D. G., "The Effect of the Earth's Oblateness on the Orbit of a Near Satellite," London Royal Society Proceedings, Section A, 247, pp.49-72. (1958).

7. Brenner, J. L., et al., "Methods for Satellite Orbit Calculation Theory and Applications," Stanford Research Institute, Menlo Park, California. SRI Project No. SU-2587. (1959)

8. Sterne, Theodore E., "The Gravitational Orbit of a Satellite of an Oblate Planet," Astronomical Journal, 63, No. 1, January, 1958, pp.28-40.

9. Garfinkel, B., "On the Motion of a Satellite of an Oblate Planet," Astronomical Journal, 63 (1958) pp. 88-96.

10. Izsak, Imre G., "A Theory of Satellite Motion About an Oblate Planet I. A Second-Order Solution of Vinti's Dynamical Problem," Smithsonian Institution Astrophysical Observatory Special Report No. 52. November, 1960.

11. Morse, Philip M., and Feshbach, Herman, Methods of Theoretical Physics. Vol. I. New York, McGraw-Hill Book Company, Inc. 1953. 997 pp.

12. Hancock, Harris, Theory of Elliptic Functions. Dover Publications, Inc. New York. (1958).

13. Byrd, Paul F., and Friedman, Morris. D., Handbook of Elliptic Integrals for Engineers and Physicists, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit Besouder Berucksechtegung der Anwedungsgebiete. Bd LXVII. Springer-Verlag, Berlin-Gottingen-Heidelberg. 1954. viii + 355 pp.

14. Radon, Brigitte, "Sviluppi in serie degli ellitici," Atti. Accad. Mag. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I(8). pp 69-109. (1950).

15. Goldstein, Herbert, Classical Mechanics, Addison-Wesley Publishing Company, Inc., London. xii + 399 pp.

16. Vinti, John P., "Mean Motions in Conditionally Periodic Separable Systems," U. S. National Bureau of Standards Report No. 6998, November, 1960.

17. Moulton, F. R., Celestial Machanics, Second Revised Edition, New York, The MacMillan Company, 1923. 437 pp.